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Chapter 5

Capacitance and Dielectrics

Capacitance and Dielectrics

5.1 Introduction

A capacitor is a device that stores electric charge. Capacitors vary in shape and size, but the basic configuration is two conductors carrying equal but opposite charges (Figure 5.1.1). Capacitors have many important applications in electronics. Some examples include storing electric potential energy, delaying voltage changes when coupled with resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency-dependent and independent voltage dividers when combined with resistors. Some of these applications will be discussed in latter chapters.

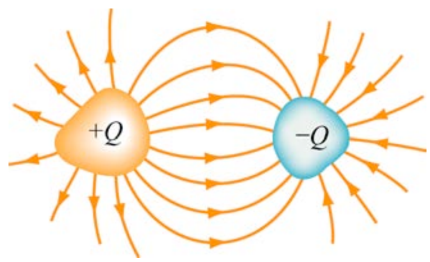


Figure 5.1.1 Basic configuration of a capacitor.

In the *uncharged* state, the charge on either one of the conductors in the capacitor is zero. During the charging process, a charge Q is moved from one conductor to the other one, giving one conductor a charge, and the other one a charge $-Q$. A potential difference ΔV is created, with the positively charged conductor at a higher potential than the negatively charged conductor. Note that whether charged or uncharged, the net charge on the capacitor as a whole is zero.

The simplest example of a capacitor consists of two conducting plates of area A, which are parallel to each other, and separated by a distance *d*, as shown in Figure 5.1.2.

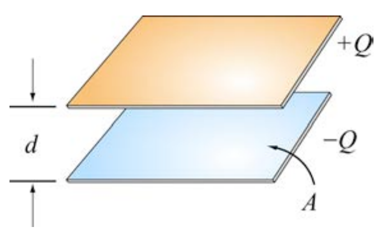


Figure 5.1.2 A parallel-plate capacitor

Experiments show that the amount of charge Q stored in a capacitor is linearly proportional to ΔV, the electric potential difference between the plates. Thus, we may write

$$Q = C\,|\Delta V|. \tag{5.1.1}$$

where C is a positive proportionality constant called *capacitance*. Physically, capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The SI unit of capacitance is the *farad* [F]:

$$1\ \text{F} = 1\ \text{farad} = 1\ \text{coulomb/volt} = 1\ \text{C/V}.$$

A typical capacitance that one finds in a laboratory is in the picofarad ($1\ \text{pF} = 10^{-12}\,\text{F}$) to millifarad range, ($1\ \text{mF} = 10^{-3}\,\text{F} = 1000\,\mu\text{F};\ 1\,\mu\text{F} = 10^{-6}\,\text{F}$).

Figure 5.1.3(a) shows the symbol that is used to represent capacitors in circuits. For a polarized fixed capacitor that has a definite polarity, Figure 5.1.3(b) is sometimes used.

Figure 5.1.3 Capacitor symbols.

5.2 Calculation of Capacitance

Let's see how capacitance can be computed in systems with simple geometry.

Example 5.1: Parallel-Plate Capacitor

Consider two metallic plates of equal area A separated by a distance d, as shown in Figure 5.2.1 below. The top plate carries a charge $+Q$ while the bottom plate carries a charge $-Q$. The charging of the plates can be accomplished by means of a battery, which produces a potential difference. Find the capacitance of the system.

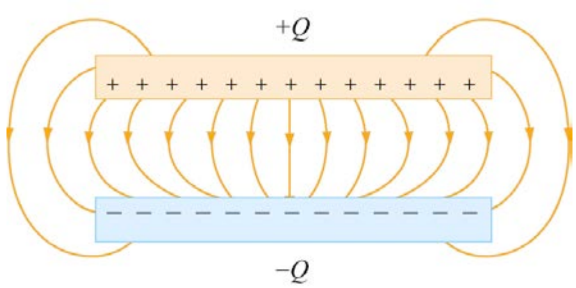


Figure 5.2.1 The electric field between the plates of a parallel-plate capacitor

Solution: To find the capacitance C, we first need to know the electric field between the plates. A real capacitor is finite in size. Thus, the electric field lines at the edge of the plates are not straight lines, and the field is not contained entirely between the plates. This is known as *edge effects*, and the non-uniform fields near the edge are called the *fringing fields*. In Figure 5.2.1, the field lines are drawn incorporating edge effects. However, in what follows, we shall ignore such effects and assume an idealized situation, where field lines between the plates are straight lines, and zero outside.

In the limit where the plates are infinitely large, the system has planar symmetry and we can calculate the electric field everywhere using Gauss's law given in Eq. (3.2.5):

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

By choosing a Gaussian "pillbox" with cap area A' to enclose the charge on the positive plate (see Figure 5.2.2), the electric field in the region between the plates is

$$EA' = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A'}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{\varepsilon_0}. \tag{5.2.1}$$

The same result has also been obtained in Section 3.8.1 using the superposition principle.

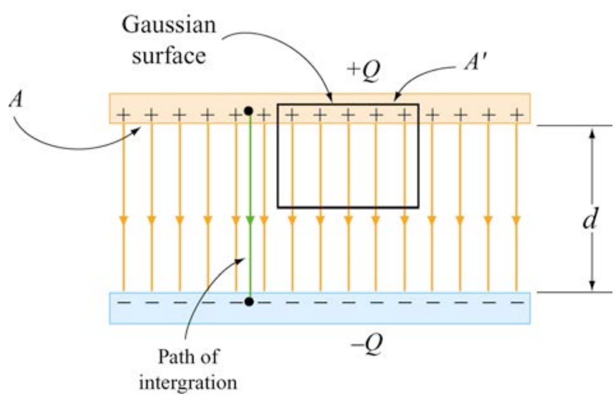


Figure 5.2.2 Gaussian surface for calculating the electric field between the plates.

The potential difference between the plates is

$$\Delta V = V_- - V_+ = -\int_+^- \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -Ed, \tag{5.2.2}$$

where we have taken the path of integration to be a straight line from the positive plate to the negative plate following the field lines (Figure 5.2.2). Because the electric field lines are always directed from higher potential to lower potential, $V_- < V_+$. However, in computing the capacitance C, the relevant quantity is the magnitude of the potential difference:

$$|\Delta V| = Ed, \tag{5.2.3}$$

and its sign is immaterial. From the definition of capacitance, we have

$$C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d} \quad \text{(parallel plate)}\,. \tag{5.2.4}$$

Note that C depends only on the geometric factors A and d. The capacitance C increases linearly with the area A since for a given potential difference ΔV, a bigger plate can hold more charge. On the other hand, C is inversely proportional to d, the distance of separation because the smaller the value of d, the smaller the potential difference $|\Delta V|$ for a fixed Q.

Example 5.2: Cylindrical Capacitor

Consider next a solid cylindrical conductor of radius a surrounded by a coaxial cylindrical shell of inner radius b, as shown in Figure 5.2.3. The length of both cylinders is L and we take this length to be much larger than $b-a$, the separation of the cylinders, so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge $+Q$ while the outer shell has a charge $-Q$. What is the capacitance?

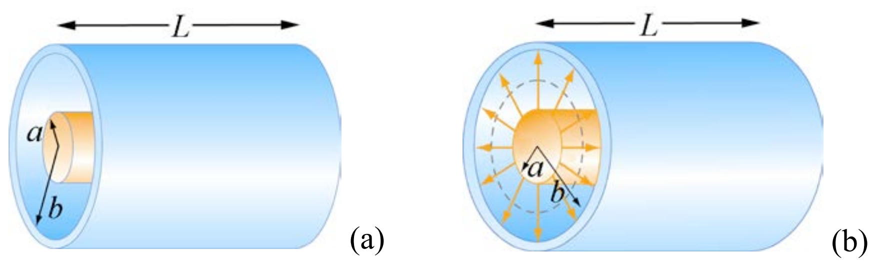


Figure 5.2.3 (a) A cylindrical capacitor. (b) End view of the capacitor. The electric field is non-vanishing only in the region $a < r < b$.

Solution:

To calculate the capacitance, we first compute the electric field everywhere. Due to the cylindrical symmetry of the system, we choose our Gaussian surface to be a coaxial cylinder with length $\ell < L$ and radius r where $a < r < b$. Using Gauss's law, we have

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E\left(2\pi r \ell\right) = \frac{\lambda \ell}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}\,, \tag{5.2.5}$$

where $\lambda = Q/L$ is the charge per unit length. Notice that the electric field is non-vanishing only in the region $a < r < b$. For $r < a$, the enclosed charge is $q_{\text{enc}} = 0$ because in electrostatic equilibrium any charge in a conductor must reside on its surface. Similarly, for $r > b$, the enclosed charge is $q_{\text{enc}} = \lambda\ell - \lambda\ell = 0$ since the Gaussian surface encloses equal but opposite charges from both conductors. The potential difference is given by

$$\Delta V = V_b - V_a = -\int_a^b E_r\, dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right), \tag{5.2.6}$$

where we have chosen the integration path to be along the direction of the electric field lines. As expected, the outer conductor with negative charge has a lower potential. The capacitance is then

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{\lambda L}{\lambda \ln(b/a)/2\pi\varepsilon_0} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}}. \tag{5.2.7}$$

Once again, we see that the capacitance C depends only on the length L, and the radii a and b.

Example 5.3: Spherical Capacitor

As a third example, let's consider a spherical capacitor which consists of two concentric spherical shells of radii a and b, as shown in Figure 5.2.4. The inner shell has a charge $+Q$ uniformly distributed over its surface, and the outer shell an equal but opposite charge $-Q$. What is the capacitance of this configuration?

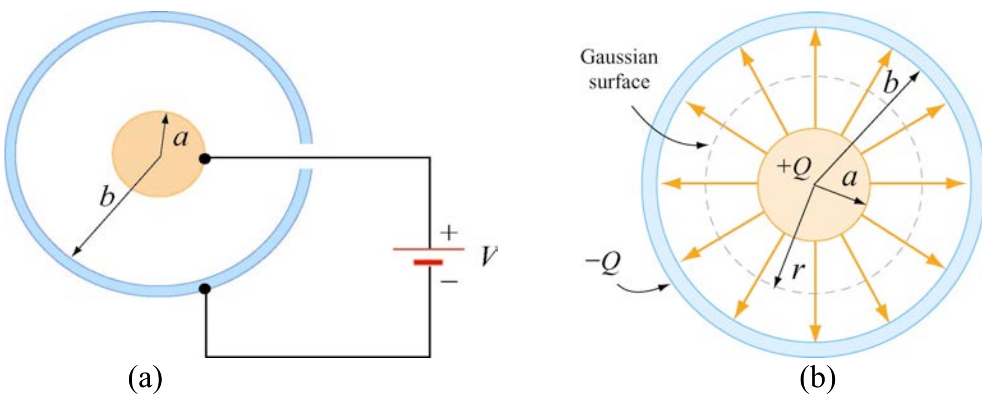


Figure 5.2.4 (a) spherical capacitor with two concentric spherical shells of radii a and b. (b) Gaussian surface for calculating the electric field.

Solution: The electric field is non-vanishing only in the region $a < r < b$. Using Gauss's law, we obtain

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_r A = E_r(4\pi r^2) = \frac{Q}{\varepsilon_0}. \tag{5.2.8}$$

The radial component of the electric field is then

$$E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}. \tag{5.2.9}$$

Therefore, the potential difference between the two conducting shells is:

$$\Delta V = V_b - V_a = -\int_a^b E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right), \qquad (5.2.10)$$

which yields for the capacitance

$$\boxed{C = \frac{Q}{|\Delta V|} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)}. \qquad (5.2.11)$$

The capacitance C depends only on the radii a and b.

An "isolated" conductor (with the second conductor placed at infinity) also has a capacitance. In the limit where $b \to \infty$, the above equation becomes

$$\lim_{b\to\infty} C = \lim_{b\to\infty} 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right) = \lim_{b\to\infty} 4\pi\varepsilon_0 \frac{a}{\left(1-\frac{a}{b}\right)} = 4\pi\varepsilon_0 a\,. \qquad (5.2.12)$$

Thus, for a single isolated spherical conductor of radius R, the capacitance is

$$C = 4\pi\varepsilon_0 R\,. \qquad (5.2.13)$$

The above expression can also be obtained by noting that a conducting sphere of radius R with a charge Q uniformly distributed over its surface has $V = Q/4\pi\varepsilon_0 R$, where infinity is the reference point at zero potential, $V(\infty)=0$. Using our definition for capacitance,

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R\,. \qquad (5.2.14)$$

As expected, the capacitance of an isolated charged sphere only depends on the radius R.

5.3 Storing Energy in a Capacitor

A capacitor can be charged by connecting the plates to the terminals of a battery, which are maintained at a potential difference ΔV called the *terminal voltage*.

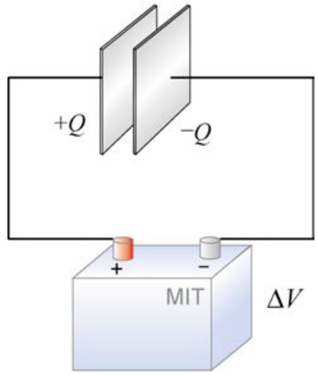


Figure 5.3.1 Charging a capacitor.

The connection results in sharing the charges between the terminals and the plates. For example, the plate that is connected to the (positive) negative terminal will acquire some (positive) negative charge. The sharing causes a momentary reduction of charges on the terminals, and a decrease in the terminal voltage. Chemical reactions are then triggered to transfer more charge from one terminal to the other to compensate for the loss of charge to the capacitor plates, and maintain the terminal voltage at its initial level. The battery could thus be thought of as a charge pump that brings a charge Q from one plate to the other.

As discussed in the introduction, capacitors can be used to stored electrical energy. The amount of energy stored is equal to the work done to charge it. During the charging process, the battery does work to remove charges from one plate and deposit them onto the other.

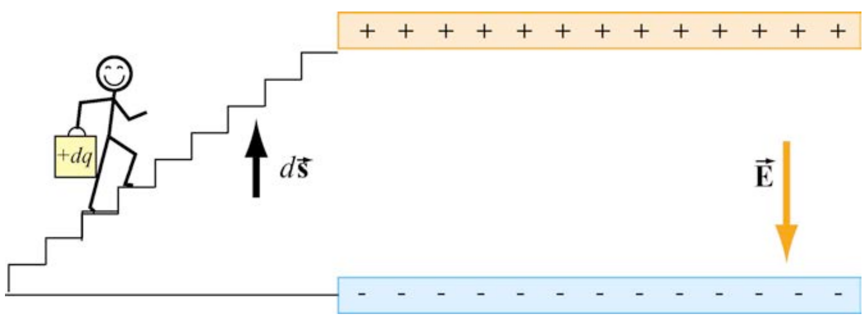


Figure 5.3.1 Work is done by an external agent in bringing $+dq$ from the negative plate and depositing the charge on the positive plate.

Let the capacitor be initially uncharged. In each plate of the capacitor, there are many negative and positive charges, but the number of negative charges balances the number of positive charges, so that there is no net charge, and therefore no electric field between the plates. We have a magic bucket and a set of stairs from the bottom plate to the top plate (Figure 5.3.1). We show a movie of what is essentially this process in Section 5.5.2 below.

We start out at the bottom plate, fill our magic bucket with a charge $+dq$, carry the bucket up the stairs and dump the contents of the bucket on the top plate, charging it up positive to charge $+dq$. However, in doing so, the bottom plate is now charged to $-dq$. Having emptied the bucket of charge, we now descend the stairs, get another bucketful of charge $+dq$, go back up the stairs and dump that charge on the top plate. We then repeat this process over and over. In this way we build up charge on the capacitor, and create electric field where there was none initially.

Suppose the amount of charge on the top plate at some instant is $+q$, and the potential difference between the two plates is $|\Delta V| = q/C$. To dump another bucket of charge $+dq$ on the top plate, the amount of work done to overcome electrical repulsion is $dW = |\Delta V|\,dq$. If at the end of the charging process, the charge on the top plate is $+Q$, then the total amount of work done in this process is

$$W = \int_0^Q dq\,|\Delta V| = \int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C}. \tag{5.3.1}$$

This is equal to the electrical potential energy U_E of the system:

$$\boxed{U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q|\Delta V| = \frac{1}{2}C|\Delta V|^2}. \tag{5.3.2}$$

5.3.1 Energy Density of the Electric Field

One can think of the energy stored in the capacitor as being stored in the electric field itself. In the case of a parallel-plate capacitor, with $C = \varepsilon_0 A/d$ and $|\Delta V| = Ed$, we have

$$U_E = \frac{1}{2}C|\Delta V|^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2(Ad). \tag{5.3.3}$$

Because the quantity Ad represents the volume between the plates, we can define the electric energy density as

$$\boxed{u_E = \frac{U_E}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2}. \tag{5.3.4}$$

The energy density u_E is proportional to the square of the electric field. Alternatively, one may obtain the energy stored in the capacitor from the point of view of external work. Because the plates are oppositely charged, force must be applied to maintain a constant separation between them. From Eq. (3.4.7), we see that a small patch of charge $\Delta q = \sigma(\Delta A)$ experiences an attractive force $\Delta F = \sigma^2(\Delta A)/2\varepsilon_0$. If the total area of the plate is A, then an external agent must exert a force $F_{\text{ext}} = \sigma^2 A/2\varepsilon_0$ to pull the two plates apart. Since the electric field strength in the region between the plates is given by $E = \sigma / \varepsilon_0$, the external force can be rewritten as

$$F_{\text{ext}} = \frac{\varepsilon_0}{2} E^2 A . \tag{5.3.5}$$

The external force F_{ext} is independent of d. The total amount of work done externally to separate the plates by a distance d is then

$$W_{\text{ext}} = \int \vec{\mathbf{F}}_{\text{ext}} \cdot d\vec{\mathbf{s}} = F_{\text{ext}} d = \left(\frac{\varepsilon_0 E^2 A}{2} \right) d , \tag{5.3.6}$$

consistent with Eq. (5.3.3). Because the potential energy of the system is equal to the work done by the external agent, we have that the energy density $u_E = W_{\text{ext}} / Ad = \varepsilon_0 E^2 / 2$. In addition, we note that the expression for u_E is identical to Eq. (3.4.8) in Chapter 3. Therefore, the electric energy density u_E can also be interpreted as electrostatic pressure P.

Example 5.4: Electric Energy Density of Dry Air

The breakdown field strength at which dry air loses its insulating ability and allows a discharge to pass through is $E_b = 3 \times 10^6 \text{ V/m}$. At this field strength, the electric energy density is:

$$u_E = \frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2} (8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(3 \times 10^6 \text{ V/m})^2 = 40 \text{ J/m}^3 . \tag{5.3.7}$$

Example 5.5: Energy Stored in a Spherical Shell

Find the energy stored in a metallic spherical shell of radius a and charge Q.

Solution: The electric field associated of a spherical shell of radius a is (Example 3.3)

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}, & r > a \\ \vec{\mathbf{0}}, & r < a. \end{cases} \tag{5.3.8}$$

The corresponding energy density is

$$u_E = \frac{1}{2} \varepsilon_0 E^2 = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} , \tag{5.3.9}$$

outside the sphere, and zero inside. Since the electric field is non-vanishing outside the spherical shell, we must integrate over the entire region of space from $r = a$ to $r = \infty$. In spherical coordinates, with $dV = 4\pi r^2 dr$, we have

$$U_E = \int_a^\infty \left(\frac{Q^2}{32\pi^2 \varepsilon_0 r^4} \right) 4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_0} \int_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 a} = \frac{1}{2} QV , \qquad (5.3.10)$$

where $V = Q/4\pi\varepsilon_0 a$ is the electric potential on the surface of the shell, with $V(\infty) = 0$. We can readily verify that the energy of the system is equal to the work done in charging the sphere. To show this, suppose at some instant the sphere has charge q and is at a potential $V = q/4\pi\varepsilon_0 a$. The work required to add an additional charge dq to the system is $dW = Vdq$. Thus, the total work is

$$W = \int dW = \int V dq = \int_0^Q dq \left(\frac{q}{4\pi\varepsilon_0 a} \right) = \frac{Q^2}{8\pi\varepsilon_0 a} . \qquad (5.3.11)$$

5.4 Dielectrics

In many capacitors there is an insulating material such as paper or plastic between the plates. Such material, called a dielectric, can be used to maintain a physical separation of the plates. Since dielectrics break down less readily than air, charge leakage can be minimized, especially when high voltage is applied.

Experimentally it was found that capacitance C increases when the space between the conductors is filled with dielectrics. To see how this happens, suppose a capacitor has a capacitance C_0 when there is no material between the plates. When a dielectric material is inserted to completely fill the space between the plates, the capacitance increases to

$$C = \kappa_e C_0 , \qquad (5.4.1)$$

where κ_e is called the dielectric constant. In the Table below, we show some dielectric materials with their dielectric constant. Experiments indicate that all dielectric materials have $\kappa_e > 1$. Note that every dielectric material has a characteristic dielectric strength that is the maximum value of electric field before breakdown occurs and charges begin to flow.

Material	κ_e	Dielectric strength $(10^6 \mathrm{V/m})$
Air	1.00059	3
Paper	3.7	16
Glass	4–6	9
Water	80	–

The increase of capacitance in the presence of a dielectric can be explained from a molecular point of view. We shall show that κ_e is a measure of the dielectric response to an external electric field. There are two types of dielectrics. The first type are polar dielectrics, which are dielectrics that have permanent electric dipole moments. An example of this type of dielectric is water.

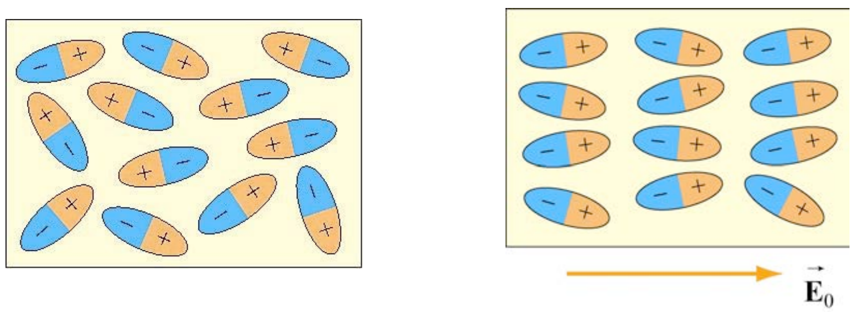


Figure 5.4.1 Orientations of polar molecules when (a) $\vec{\mathbf{E}}_0 = \vec{\mathbf{0}}$ and (b) $\vec{\mathbf{E}}_0 \neq 0$.

As depicted in Figure 5.4.1, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\vec{\mathbf{E}}_0$ is present, a torque is set up that causes the molecules to align with $\vec{\mathbf{E}}_0$. However, the alignment is not complete due to random thermal motion. The aligned molecules then generate an electric field that is opposite to the applied field but smaller in magnitude.

The second type are non-polar dielectrics, which are dielectrics that do not possess a permanent electric dipole moment. Placing a non-polar dielectric material in an externally applied electric field can induce electric dipole moments.

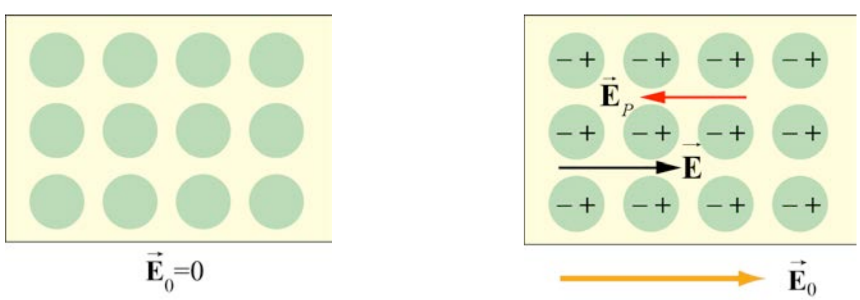


Figure 5.4.2 Orientations of non-polar molecules when (a) $\vec{\mathbf{E}}_0 = \vec{\mathbf{0}}$ and (b) $\vec{\mathbf{E}}_0 \neq \vec{\mathbf{0}}$.

Figure 5.4.2 illustrates the orientation of non-polar molecules with and without an external field $\vec{\mathbf{E}}_0$. When $\vec{\mathbf{E}}_0 \neq \vec{\mathbf{0}}$, (Figure 5.4.2(b)), the induced surface charges on the faces produces an electric field $\vec{\mathbf{E}}_P$ in the direction opposite to $\vec{\mathbf{E}}_0$, leading to $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P$, with $|\vec{\mathbf{E}}| < |\vec{\mathbf{E}}_0|$. Below we show how the induced electric field $\vec{\mathbf{E}}_P$ is calculated.

5.4.1 Polarization

We have shown that dielectric materials consist of many permanent or induced electric dipoles. One of the concepts crucial to the understanding of dielectric materials is the average electric field produced by many little electric dipoles that are all aligned. Suppose we have a piece of material in the form of a cylinder with area A and height h, as shown in Figure 5.4.3, and that it consists of N electric dipoles, each with electric dipole moment $\vec{\mathbf{p}}$ spread uniformly throughout the volume of the cylinder.

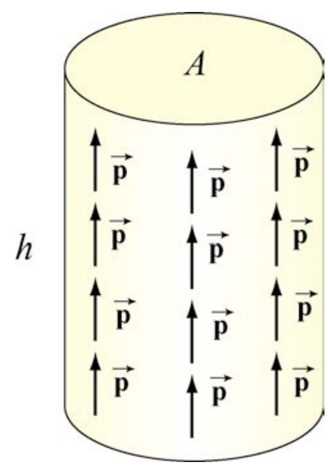


Figure 5.4.3 A cylinder with uniform dipole distribution.

We furthermore assume for the moment that all of the electric dipole moments $\vec{\mathbf{p}}$ are aligned with the axis of the cylinder. Since each electric dipole has its own electric field associated with it, in the absence of any external electric field, if we average over all the individual fields produced by the dipole, what is the average electric field just due to the presence of the aligned dipoles?

To answer this question, let us define the polarization vector $\vec{\mathbf{P}}$ to be the net electric dipole moment vector per unit volume:

$$\vec{\mathbf{P}} = \frac{1}{\text{Volume}} \sum_{i=1}^{N} \vec{\mathbf{p}}_i \,. \tag{5.4.2}$$

In the case of our cylinder, where all the dipoles are perfectly aligned, the magnitude of $\vec{\mathbf{P}}$ is equal to

$$P = \frac{Np}{Ah}. \tag{5.4.3}$$

The direction of $\vec{\mathbf{P}}$ is parallel to the aligned dipoles.

Now, what is the average electric field these dipoles produce? All the little ± charges associated with the electric dipoles in the interior of the cylinder in Figure 5.4.4(a) are replaced by two equivalent charges, $\pm Q_P$, on the top and bottom of the cylinder, respectively in Figure 5.4.4(b).

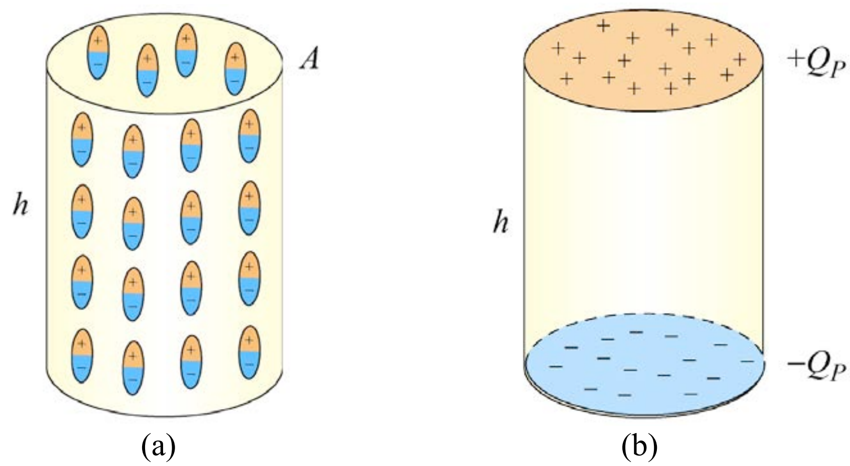


Figure 5.4.4 (a) A cylinder with uniform dipole distribution. (b) Equivalent charge distribution.

The equivalence can be seen by noting that in the interior of the cylinder, positive charge at the top of any one of the electric dipoles is *canceled* on average by the negative charge of the dipole just above it. The only places where cancellations do not take place are at the top and bottom of the cylinder, where there are no additional adjacent dipoles. Thus the interior of the cylinder appears uncharged in an average sense (averaging over many dipoles). The top surface of the cylinder carries a positive charge and the bottom surface of the cylinder carries a negative charge.

How do we find an expression for the equivalent charge Q_P in terms of quantities we know? The simplest way is to require that the electric dipole moment Q_P produces, $Q_P h$, is equal to the total electric dipole moment of all the little electric dipoles. This gives $Q_P h = Np$, hence

$$Q_P = \frac{Np}{h}. \tag{5.4.4}$$

To compute the electric field produced by Q_P, we note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_P that is equal to the magnitude of the polarization:

$$\sigma_P = \frac{Q_P}{A} = \frac{Np}{Ah} = P. \tag{5.4.5}$$

The SI units of polarization density, P, are $(\mathrm{C}\cdot\mathrm{m})/\mathrm{m}^3$, or $\mathrm{C/m}^2$, which are the same units as surface charge density. In general if the polarization vector makes an angle θ with $\hat{\mathbf{n}}$, the outward normal vector of the surface, the surface charge density would be

$$\sigma_P = \vec{\mathbf{P}}\cdot\hat{\mathbf{n}} = P\cos\theta. \tag{5.4.6}$$

The equivalent charge system will produce an average electric field of magnitude $E_P = P/\varepsilon_0$. Because the direction of this electric field is *opposite* to the direction of $\vec{\mathbf{P}}$, in vector notation, we have

$$\vec{\mathbf{E}}_P = -\vec{\mathbf{P}}/\varepsilon_0. \tag{5.4.7}$$

The average electric field of all these dipoles is opposite to the direction of the dipoles themselves. It is important to realize that this is just the *average* field due to all the dipoles. If we go close to any individual dipole, we will see a very different field.

We have assumed here that all our electric dipoles are aligned. In general, if these dipoles are randomly oriented, then the polarization $\vec{\mathbf{P}}$ given in Eq. (5.4.2) will be zero, and there will be no average field due to their presence. If the dipoles have some tendency toward a preferred orientation, then $\vec{\mathbf{P}} \neq \vec{\mathbf{0}}$, leading to a non-vanishing average field $\vec{\mathbf{E}}_P$.

Let us now examine the effects of introducing a dielectric material into a system. We shall first assume that the atoms or molecules comprising the dielectric material have a *permanent* electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field, $\vec{\mathbf{P}} = \vec{\mathbf{0}}$ due to the random alignment of dipoles, and the average electric field $\vec{\mathbf{E}}_P$ is zero as well. However, when we place the dielectric material in an external field $\vec{\mathbf{E}}_0$, the dipoles will experience a torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}}\times\vec{\mathbf{E}}_0$ that tends to align the dipole vectors $\vec{\mathbf{p}}$ with $\vec{\mathbf{E}}_0$. The effect is a net polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and therefore the dipoles produce an average electric field, $\vec{\mathbf{E}}_P$, *anti-parallel* to $\vec{\mathbf{E}}_0$, i.e., that will tend to *reduce* the total electric field strength below $\left|\vec{\mathbf{E}}_0\right|$. The electric field $\vec{\mathbf{E}}$ is the sum of these two fields:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0 - \vec{\mathbf{P}}/\varepsilon_0. \tag{5.4.8}$$

In most cases, the polarization $\vec{\mathbf{P}}$ is not only in the same direction as $\vec{\mathbf{E}}_0$, but also linearly proportional to $\vec{\mathbf{E}}_0$, and hence to $\vec{\mathbf{E}}$ as well. This is reasonable because without the external field $\vec{\mathbf{E}}_0$ there would be no alignment of dipoles and no polarization $\vec{\mathbf{P}}$. We write the linear relation between $\vec{\mathbf{P}}$ and $\vec{\mathbf{E}}$ as

$$\vec{\mathbf{P}} = \varepsilon_0 \chi_e \vec{\mathbf{E}}, \tag{5.4.9}$$

where χ_e is called the *electric susceptibility*. Materials that obey this relation are called linear *dielectrics*. Combing Eqs. (5.4.8) and (5.4.7) yields

$$\vec{\mathbf{E}}_0 = (1+\chi_e)\vec{\mathbf{E}} = \kappa_e \vec{\mathbf{E}}, \tag{5.4.10}$$

where

$$\kappa_e = (1+\chi_e) \tag{5.4.11}$$

is the dielectric constant. The dielectric constant κ_e is always greater than one since $\chi_e > 0$. This implies that

$$E = \frac{E_0}{\kappa_e} < E_0 \,. \tag{5.4.12}$$

Thus, we see that the effect of dielectric materials is always to decrease the electric field below what it would otherwise be.

In the case of dielectric material where there are no permanent electric dipoles, a similar effect is observed because the presence of an external field $\vec{\mathbf{E}}_0$ induces electric dipole moments in the atoms or molecules. These induced electric dipoles are parallel to $\vec{\mathbf{E}}_0$, again leading to a polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and a reduction of the total electric field strength.

5.4.2 Dielectrics without Battery

As shown in Figure 5.4.5, a battery with a potential difference $|\Delta V_0|$ across its terminals is first connected to a capacitor C_0, which holds a charge $Q_0 = C_0 |\Delta V_0|$. We then disconnect the battery, leaving Q_0. The charge Q_0 is called the *free charge* and when the battery is disconnected does not change (because it has no conducting path off the plate).

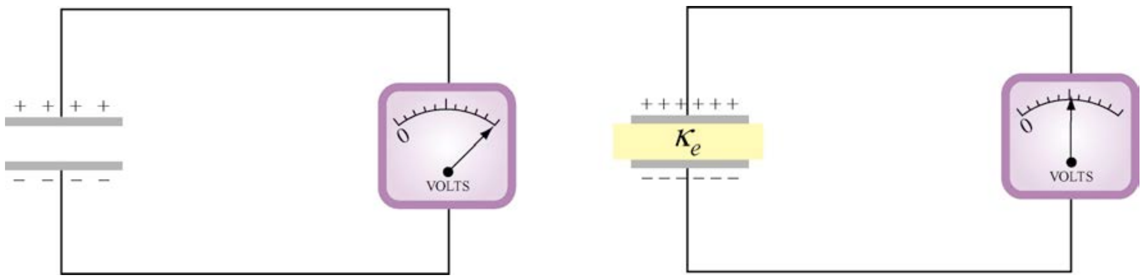


Figure 5.4.5 Inserting a dielectric material between the capacitor plates while keeping the charge Q_0 constant

If we then insert a dielectric between the plates (while keeping the free charge constant), experimentally it is found that the potential difference decreases by a factor of κ_e:

$$|\Delta V| = \frac{|\Delta V_0|}{\kappa_e}. \tag{5.4.13}$$

This implies that the capacitance is changed to

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{Q_0}{|\Delta V_0|/\kappa_e} = \kappa_e \frac{Q_0}{|\Delta V_0|} = \kappa_e C_0}. \tag{5.4.14}$$

The capacitance has increased by a factor of κ_e. The electric field within the dielectric is now

$$E = \frac{|\Delta V|}{d} = \frac{|\Delta V_0|/\kappa_e}{d} = \frac{1}{\kappa_e}\left(\frac{|\Delta V_0|}{d}\right) = \frac{E_0}{\kappa_e}. \tag{5.4.15}$$

In the presence of a dielectric, the electric field decreases by a factor of κ_e.

5.4.3 Dielectrics with Battery

Consider a second case where a battery supplying a potential difference $|\Delta V_0|$ remains connected as the dielectric is inserted (Figure 5.4.6). Experimentally, it is found (first by Faraday) that the charge on the plates is increased by a factor κ_e:

$$Q = \kappa_e Q_0, \tag{5.4.16}$$

where Q_0 is the free charge on the plates in the absence of any dielectric.

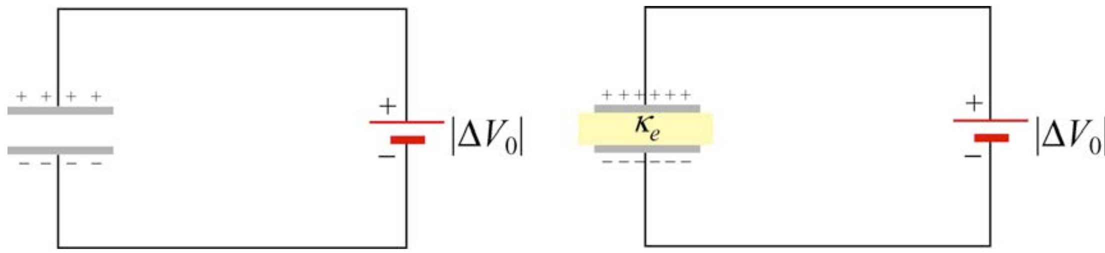


Figure 5.4.6 Inserting a dielectric material between the capacitor plates while maintaining a constant potential difference $|\Delta V_0|$.

The capacitance becomes

$$\boxed{C = \frac{Q}{|\Delta V_0|} = \frac{\kappa_e Q_0}{|\Delta V_0|} = \kappa_e C_0} \tag{5.4.17}$$

increasing because the battery has delivered more free charge to the plates resulting in the magnitude of the charge on either plate increasing.

In either case, the new value of the capacitance does not depend on whether or not the battery is connected while the dielectric material is inserted. However, the electric field, and charge on the plates do depend on whether or not the battery was connected while the dielectric was inserted.

5.4.4 Gauss's Law for Dielectrics

Consider again a parallel-plate capacitor shown in Figure 5.4.7:

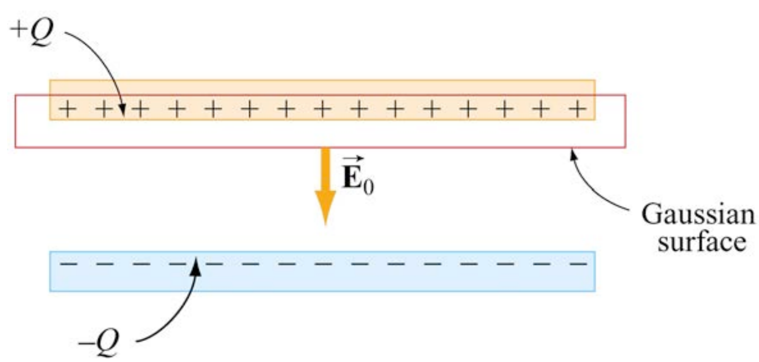


Figure 5.4.7 Gaussian surface in the absence of a dielectric.

When no dielectric is present, the electric field $\vec{\mathbf{E}}_0$ in the region between the plates can be found by using Gauss's law:

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_0 A = \frac{Q}{\varepsilon_0}, \quad \Rightarrow \quad E_0 = \frac{Q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0}.$$

With capacitance

$$C_0 = \frac{Q}{|\Delta V|} = \frac{Q}{E_0 d} = \frac{A\varepsilon_0}{d}. \tag{5.4.18}$$

We have seen that when a dielectric is inserted (Figure 5.4.8), the capacitance increases by an amount

$$C = \kappa_e C_0 = \frac{\kappa_e Q}{|\Delta V|} = \frac{\kappa_e Q}{E_0 d} = \frac{\kappa_e A\varepsilon_0}{d}. \tag{5.4.19}$$

There is now an induced charge Q_P of opposite sign on the surface, and the net charge enclosed by the Gaussian surface is $Q - Q_P$.

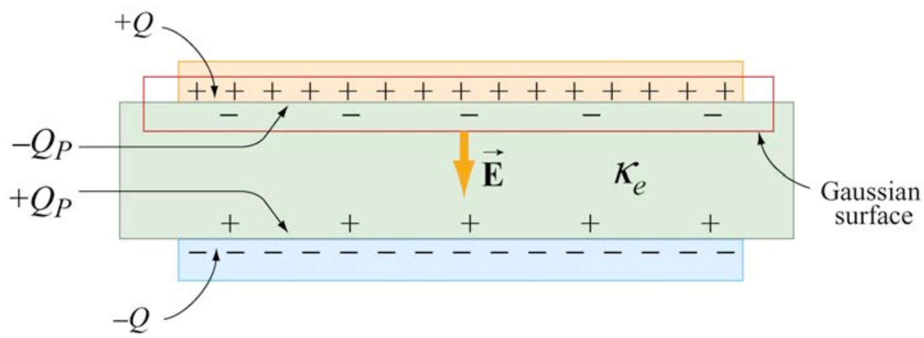


Figure 5.4.8 Gaussian surface in the presence of a dielectric.

Gauss's law becomes

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q - Q_P}{\varepsilon_0}. \tag{5.4.20}$$

The magnitude of the electric field has decreased between the plates

$$\boxed{E = \frac{Q - Q_P}{\varepsilon_0 A}}. \tag{5.4.21}$$

However, we have just seen that the effect of the dielectric is to weaken the original field E_0 by a factor κ_e. Therefore,

$$E = \frac{E_0}{\kappa_e} = \frac{Q}{\kappa_e \varepsilon_0 A} = \frac{Q - Q_P}{\varepsilon_0 A}. \tag{5.4.22}$$

from which the induced charge Q_P can be obtained as

$$Q_P = Q\left(1 - \frac{1}{\kappa_e}\right). \tag{5.4.23}$$

In terms of the surface charge density (divide Eq. (5.4.23)) by the area of the plate, we have

$$\sigma_P = \sigma\left(1 - \frac{1}{\kappa_e}\right). \tag{5.4.24}$$

The limit as $\kappa_e = 1$, the induced charge is zero, $Q_P = 0$, which corresponds to the case of no dielectric material. Substituting Eq. (5.4.23) into Eq. (5.4.20), Gauss's law with dielectric can be rewritten as,

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{free,enc}}}{\kappa_e \varepsilon_0} = \frac{Q_{\text{free,enc}}}{\varepsilon}, \tag{5.4.25}$$

where $Q_{\text{free,enc}}$ is the free charge enclosed and $\varepsilon = \kappa_e \varepsilon_0$ is called the *dielectric permittivity*. Alternatively, we may also write

$$\oint\!\!\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{A}} = Q_{\text{free,enc}}, \tag{5.4.26}$$

where $\vec{\mathbf{D}} = \varepsilon_0 \kappa \vec{\mathbf{E}}$ is called the *electric displacement vector*.

Example 5.6: Capacitance with Dielectrics

A non-conducting slab of thickness *t*, area *A* and dielectric constant κ_e is inserted into the space between the plates of a parallel-plate capacitor with spacing *d*, charge *Q* and area *A*, as shown in Figure 5.4.9(a). The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

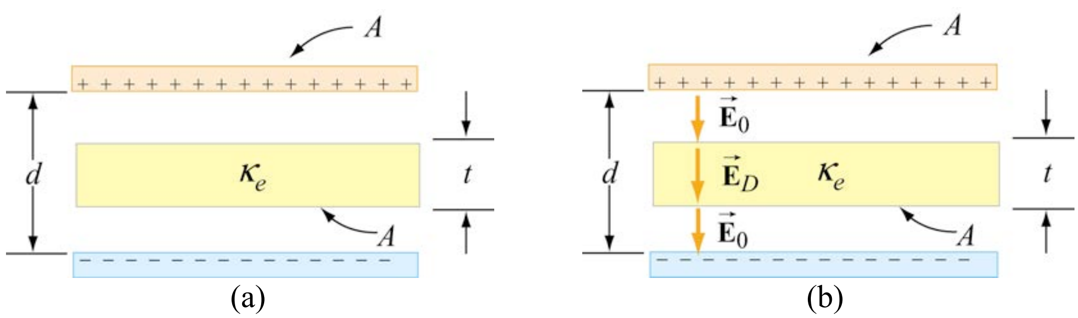


Figure 5.4.9 (a) Capacitor with a dielectric. (b) Electric field between the plates.

Solution: To find the capacitance *C*, we first calculate the potential difference ΔV. We have already seen that in the absence of a dielectric, the electric field between the plates is given by $E_0 = Q/\varepsilon_0 A$, and $E_D = E_0/\kappa_e$ when a dielectric of dielectric constant κ_e is present, as shown in Figure 5.4.9(b). The potential can be found by integrating the electric field along a straight line from the top to the bottom plates:

$$\begin{aligned}\Delta V &= -\int_{+}^{-} E\,dl = -\Delta V_0 - \Delta V_D = -E_0(d-t) - E_D t = -\frac{Q}{A\varepsilon_0}(d-t) - \frac{Q}{A\varepsilon_0\kappa_e}t \\ &= -\frac{Q}{A\varepsilon_0}\left[d - t\left(1-\frac{1}{\kappa_e}\right)\right],\end{aligned} \tag{5.4.27}$$

where $\Delta V_D = E_D t$ is the potential difference between the two faces of the dielectric. The capacitance is

$$C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d - t\left(1-\dfrac{1}{\kappa_e}\right)}. \tag{5.4.28}$$

It is useful to check the following limits:

(i) As $t \to 0$, *i.e.*, the thickness of the dielectric approaches zero, we have $C = \varepsilon_0 A/d = C_0$, which is the expected result for no dielectric.

(ii) As $\kappa_e \to 1$, we again have $C \to \varepsilon_0 A/d = C_0$, and the situation also correspond to the case where the dielectric is absent.

(iii) In the limit where $t \to d$, the space is filled with dielectric, we have $C \to \kappa_e \varepsilon_0 A/d = \kappa_e C_0$.

5.5 Creating Electric Fields

5.5.1 Creating an Electric Dipole Movie

Electric fields are created by electric charge. If there is no electric charge present, and never had been any electric charge present in the past, then there would be no electric field anywhere is space. How is electric field created and how does it come to fill up space? To answer this, consider the following scenario in which we go from the electric field being zero everywhere in space to an electric field existing everywhere in space.

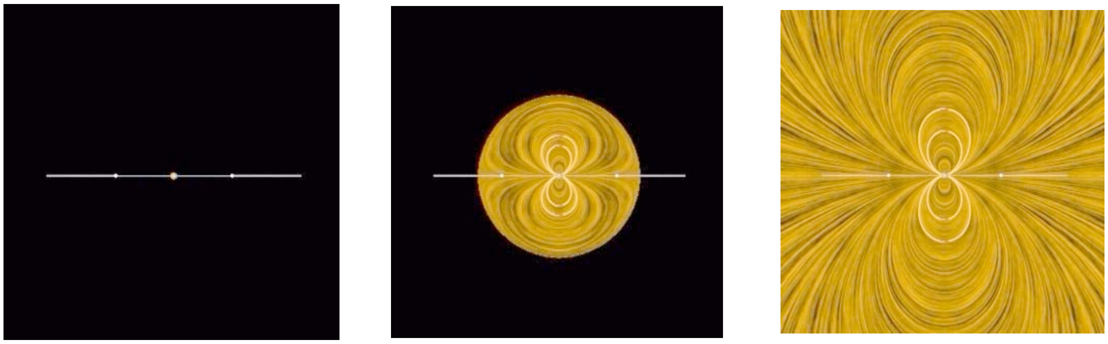


Figure 5.5.1 Creating an electric dipole. (a) Before any charge separation. (b) Just after the charges are separated. (c) A long time after separation. http://youtu.be/zIIQNZ9OAF0 .

Suppose we have a positive point charge sitting right on top of a negative electric charge, so that the total charge exactly cancels, and there is no electric field anywhere in space. Now let us pull these two charges apart slightly, so that a small distance separates them. If we allow them to sit at that distance for a long time, there will now be a charge imbalance – an electric dipole. The dipole will create an electric field.

Let us see how this creation of electric field takes place in detail. Figure 5.5.1 shows three frames of a movie of the process of separating the charges. In Figure 5.5.1(a), there is no charge separation, and the electric field is zero everywhere in space. Figure 5.5.1(b) shows what happens just after the charges are first separated. An expanding sphere of electric fields is observed. Figure 5.5.1(c) shows a long time after the charges are separated (that is, they have been at a constant distance from each other for a long time). An electric dipole has been created.

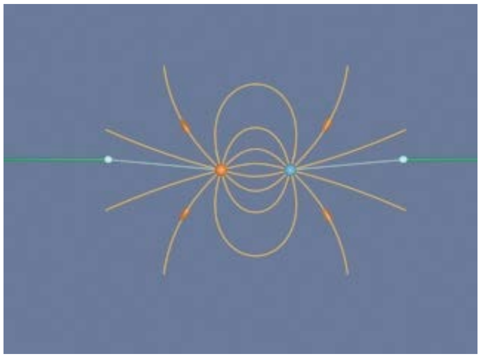
What does this sequence tell us? The following conclusions can be drawn:

(1) Electric charge generates electric field — no charge, no field.

(2) The electric field does not appear instantaneously in space everywhere as soon as there is unbalanced charge — the electric field propagates outward from its source at some finite speed. This speed will turn out to be the speed of light, as we shall see later.

(3) After the charge distribution settles down and becomes stationary, so does the field configuration. The initial field pattern associated with the time dependent separation of the charge is actually a burst of "electric dipole radiation." We return to the subject of radiation at the end of this course. Until then, we will neglect radiation fields. The field configuration left behind after a long time is just the electric dipole pattern discussed above.

We note that the external agent does work when pulling the charges apart, and then must apply a force to keep them separate, since they attract each other as soon as they start to separate. In addition, the work also goes into providing the energy carried off by radiation, as well as the energy needed to set up the final stationary electric field that we see in Figure 5.5.1(c).



http://youtu.be/CIGLshVujjM

Figure 5.5.2 Creating the electric fields of two point charges by pulling apart two opposite charges initially on top of one another. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion.

Finally, we ignore radiation and complete the process of separating our opposite point charges that we began in Figure 5.5.1. The link in Figure 5.5.2 shows the complete sequence. When we finish and have moved the charges far apart, we see the characteristic radial field in the vicinity of a point charge.

5.5.2 Creating and Destroying Electric Energy Movie

Let us look at the process of creating electric energy in a different context. We ignore energy losses due to radiation in this discussion. Figure 5.5.3 shows one frame of a movie that illustrates the following process. This movie is more or less analogous to the process we discussed in Section 5.3 above for charging a capacitor.

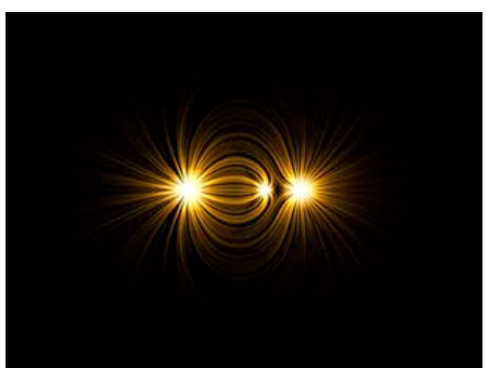


Figure 5.5.3 Creating (http://youtu.be/O5fHvc4Edvg) and destroying (http://youtu.be/5G7j0d88NGc) electric energy.

We start out with five negative electric charges and five positive charges, all at the same point in space. Sine there is no net charge, there is no electric field. Now we move one of the positive charges at constant velocity from its initial position to a distance L away along the horizontal axis. After doing that, we move the second positive charge in the same manner to the position where the first positive charge sits. After doing that, we continue on with the rest of the positive charges in the same manner, until all the positive charges are sitting a distance L from their initial position along the horizontal axis. Figure 5.5.3 shows the field configuration during this process. We have color coded the "grass seeds" representation to represent the strength of the electric field. Very strong fields are white, very weak fields are black, and fields of intermediate strength are yellow.

Over the course of the "create" movie associated with Figure 5.5.3, the strength of the electric field grows as each positive charge is moved into place. The electric energy flows out from the path along which the charges move, and is being provided by the agent moving the charge against the electric field of the other charges. The work that this agent does to separate the charges against their electric attraction appears as energy in the electric field. We also have a movie of the opposite process linked to Figure 5.5.3. That is, we return in sequence each of the five positive charges to their original positions. At the end of this process we no longer have an electric field, because we no longer have an unbalanced electric charge.

On the other hand, over the course of the "destroy" movie associated with Figure 5.5.3, the strength of the electric field decreases as each positive charge is returned to its original position. The energy flows from the field back to the path along which the charges move, and is now being provided *to* the agent moving the charge at constant speed along the electric field of the other charges. The energy provided to that agent as we destroy the electric field is exactly the amount of energy that the agent put into creating the electric field in the first place, neglecting radiative losses (such losses are small if we move the charges at speeds small compared to the speed of light). This is a reversible process if we neglect such losses. That is, the amount of energy the agent puts into creating the electric field is exactly returned to that agent as the field is destroyed.

There is one final point to be made. Whenever electromagnetic energy is being created, an electric charge is moving (or being moved) against an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}<0$). Whenever electromagnetic energy is being destroyed, an electric charge is moving (or being moved) along an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}>0$). When we return to the creation and destruction of magnetic energy, we will find this rule holds there as well.

5.6 Summary

- A **capacitor** is a device that stores electric charge and potential energy. The **capacitance** C of a capacitor is the ratio of the charge stored on the capacitor plates to the potential difference between them:

$$C=\frac{Q}{|\Delta V|}.$$

System	Capacitance
Isolated charged sphere of radius R	$C = 4\pi\varepsilon_0 R$
Parallel-plate capacitor of plate area A and plate separation d	$C = \varepsilon_0 \dfrac{A}{d}$
Cylindrical capacitor of length L, inner radius a and outer radius b	$C = \dfrac{2\pi\varepsilon_0 L}{\ln(b/a)}$
Spherical capacitor with inner radius a and outer radius b	$C = 4\pi\varepsilon_0 \dfrac{ab}{(b-a)}$

- The work done in charging a capacitor to a charge Q is

$$U = \frac{Q^2}{2C} = \frac{1}{2}Q\,|\Delta V| = \frac{1}{2}C\,|\Delta V|^2 .$$

  This is equal to the amount of energy stored in the capacitor.

- The electric energy can also be thought of as stored in the electric field $\vec{\mathbf{E}}$. The **energy density** (energy per unit volume) is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 .$$

  The energy density u_E is equal to the **electrostatic pressure** on a surface.

- When a dielectric material with **dielectric constant** κ_e is inserted into a capacitor, the capacitance increases by a factor κ_e:

$$C = \kappa_e C_0 .$$

- The **polarization** vector $\vec{\mathbf{P}}$ is the electric dipole moment per unit volume:

$$\vec{\mathbf{P}} = \frac{1}{V}\sum_{i=1}^{N} \vec{\mathbf{p}}_i .$$

- The induced electric field due to polarization is

$$\vec{\mathbf{E}}_P = -\vec{\mathbf{P}}/\varepsilon_0 .$$

- In the presence of a dielectric with dielectric constant κ_e, the electric field becomes

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0 / \kappa_e ,$$

  where $\vec{\mathbf{E}}_0$ is the electric field without dielectric.

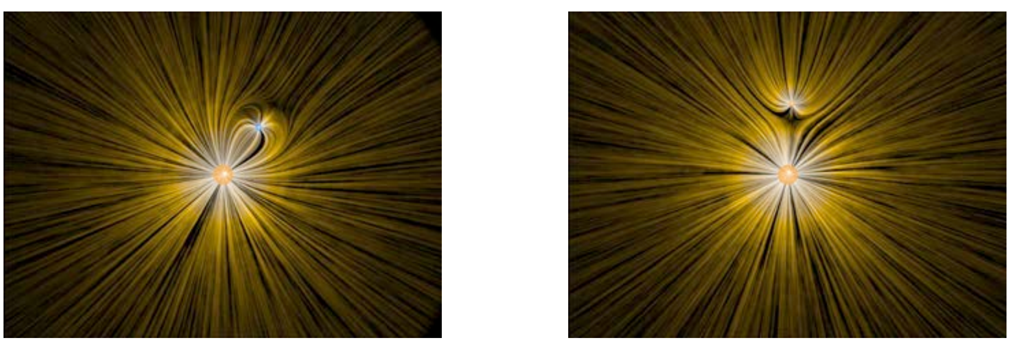
5.7 Appendix: Electric Fields Hold Atoms Together

In this Appendix, we illustrate how electric fields are responsible for holding atoms together.

> "…As our mental eye penetrates into smaller and smaller distances and shorter and shorter times, we find nature behaving so entirely differently from what we observe in visible and palpable bodies of our surroundings that no model shaped after our large-scale experiences can ever be "true". A completely satisfactory model of this type is not only practically inaccessible, but not even thinkable. Or, to be precise, we can, of course, think of it, but however we think it, it is wrong."
>
> Erwin Schroedinger

5.7.1 Ionic and van der Waals Forces

Electromagnetic forces provide the "glue" that holds atoms together—that is, that keep electrons near protons and bind atoms together in solids. We present here a brief and very idealized model of how that happens from a semi-classical point of view.



(a) http://youtu.be/C1r9-56vbio (b) http://youtu.be/pNgFql43OvM

Figure 5.7.1 (a) A negative charge and (b) a positive charge move past a massive positive particle at the origin and is deflected from its path by the stresses transmitted by the electric fields surrounding the charges.

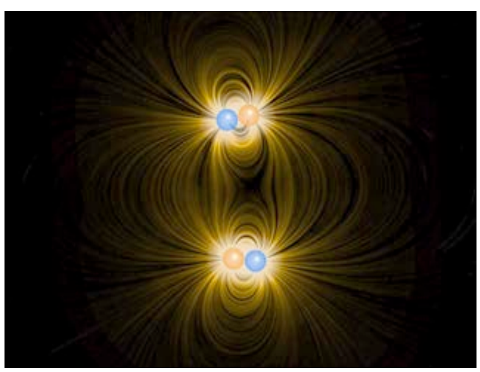
Figure 5.7.1(a) illustrates the examples of the stresses transmitted by fields, as we have seen before. In Figure 5.7.1(a) we have a negative charge moving past a massive positive charge and being deflected toward that charge due to the attraction that the two charges feel. This attraction is mediated by the stresses transmitted by the electromagnetic field, and the simple interpretation of the interaction shown in Figure 5.7.1(b) is that the attraction is primarily due to a tension transmitted by the electric fields surrounding the charges.

In Figure 5.7.1(b) we have a positive charge moving past a massive positive charge and being deflected away from that charge due to the repulsion that the two charges feel. This repulsion is mediated by the stresses transmitted by the electromagnetic field, as we have discussed above, and the simple interpretation of the interaction shown in Figure 5.7.1(b) is that the repulsion is primarily due to a pressure transmitted by the electric fields surrounding the charges.

Consider the interaction of four charges of equal mass shown in Figure 5.7.2. Two of the charges are positively charged and two of the charges are negatively charged, and all have the same magnitude of charge. The particles interact via the Coulomb force.

We also introduce a quantum-mechanical "Pauli" force, which is always repulsive and becomes very important at small distances, but is negligible at large distances. The critical distance at which this repulsive force begins to dominate is about the radius of the spheres shown in Figure 5.7.2. This Pauli force is quantum mechanical in origin, and keeps the charges from collapsing into a point (i.e., it keeps a negative particle and a positive particle from sitting exactly on top of one another).

Additionally, the motion of the particles is damped by a term proportional to their velocity, allowing them to "settle down" into stable (or meta-stable) states.



http://youtu.be/EMj10YIjkaY

Figure 5.7.2 Four charges interacting via the Coulomb force, a repulsive Pauli force at close distances, with damping.

When these charges are allowed to evolve from the initial state, the first thing that happens (very quickly) is that the charges pair off into dipoles. This is a rapid process because the Coulomb attraction between unbalanced charges is very large. This process is called "ionic binding", and is responsible for the inter-atomic forces in ordinary table salt, NaCl. After the dipoles form, there is still an interaction between neighboring dipoles, but this is a much weaker interaction because the electric field of the dipoles falls off much faster than that of a single charge. This is because the net charge of the dipole is zero. When two opposite charges are close to one another, their electric fields "almost" cancel each other out.

Although in principle the dipole-dipole interaction can be either repulsive or attractive, in practice there is a torque that rotates the dipoles so that the dipole-dipole force is attractive. After a long time, this dipole-dipole attraction brings the two dipoles together in a bound state. The force of attraction between two dipoles is termed a "van der Waals" force, and it is responsible for intermolecular forces that bind some substances together into a solid.

5.8 Problem-Solving Strategy: Calculating Capacitance

In this chapter, we have seen how capacitance C can be calculated for various systems. The procedure is summarized below:

(1) Identify the direction of the electric field using symmetry.

(2) Calculate the electric field everywhere.

(3) Compute the electric potential difference ΔV.

(4) Calculate the capacitance C using $C = Q / |\Delta V|$.

In the Table below, we illustrate how the above steps are used to calculate the capacitance of a parallel-plate capacitor, cylindrical capacitor and a spherical capacitor.

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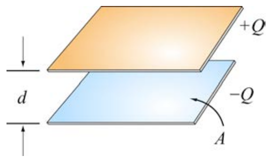
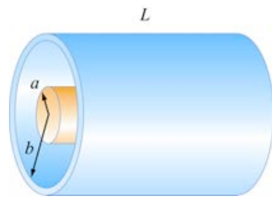
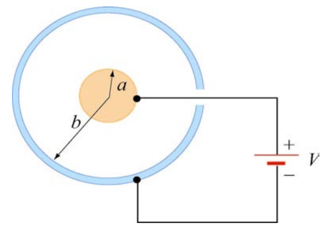
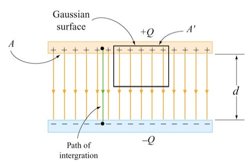
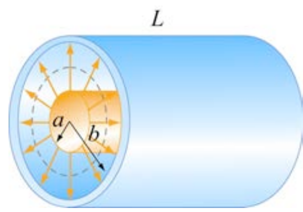
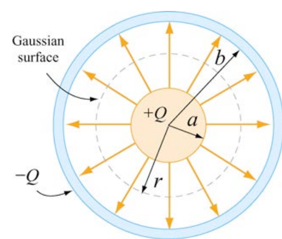
<tr><td>Figure</td><td></td><td></td><td></td></tr>
<tr><td>(1) Identify the direction of the electric field using symmetry</td><td></td><td></td><td></td></tr>
<tr><td>(2) Calculate electric field everywhere</td><td>$$\oint\!\!\oint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = EA = \frac{Q}{\varepsilon_0}$$ $$E = \frac{Q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$</td><td>$$\oint\!\!\oint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = E\left(2\pi r l\right) = \frac{Q}{\varepsilon_0}$$ $$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$</td><td>$$\oint\!\!\oint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = E_r\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0}$$ $$E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$</td></tr>
<tr><td>(3) Compute the electric potential difference ΔV</td><td>$$\Delta V = V_- - V_+ = -\int_+^- \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}$$ $$= -Ed$$</td><td>$$\Delta V = V_b - V_a = -\int_a^b E_r dr$$ $$= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$</td><td>$$\Delta V = V_b - V_a = -\int_a^b E_r dr$$ $$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$</td></tr>
<tr><td>(4) Calculate C using $C = Q/\lvert\Delta V\rvert$</td><td>$$C = \frac{\varepsilon_0 A}{d}$$</td><td>$$C = \frac{2\pi\varepsilon_0 l}{\ln(b/a)}$$</td><td>$$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$</td></tr>
</table>

5.9 Solved Problems

5.9.1 Capacitor Filled with Two Different Dielectrics

Two dielectrics with dielectric constants κ_1 and κ_2 each fill half the space between the plates of a parallel-plate capacitor as shown in Figure 5.9.1. Each plate has an area A and the plates are separated by a distance d. Compute the capacitance of the system.

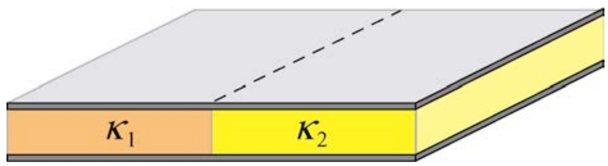


Figure 5.9.1 Capacitor filled with two different dielectrics.

Solution: Because the potential difference ΔV on each half of the capacitor is the same, we may treat the system as being composed of two capacitors, C_1 and C_2, with charges $\pm Q_1$ and $\pm Q_2$ on each half. The magnitude of the electric field is the same on each side because

$$E = \frac{|\Delta V|}{d}.$$

We can apply Eq. (5.4.25) to determine the charge on each plate in terms of the electric field between the plates:

$$Q_i = \kappa_i \varepsilon_0 E(A/2).$$

Therefore using our result for electric field, the charge is given by

$$Q_i = \frac{\kappa_i \varepsilon_0 (A/2)\,|\Delta V|}{d}.$$

The capacitance of the system is then

$$C = \frac{Q_1 + Q_2}{|\Delta V|} = \frac{\varepsilon_0 A}{2d}(\kappa_1 + \kappa_2) = C_1 + C_2,$$

where

$$C_i = \frac{\kappa_i \varepsilon_0 (A/2)}{d}, \quad i = 1, 2.$$

5.9.2 Capacitor with Dielectrics

Consider a conducting spherical shell with an inner radius a and outer radius c. Let the space between two surfaces be filed with two different dielectric materials so that the dielectric constant is κ_1 between a and b, and κ_2 between b and c, as shown in Figure 5.9.4. Determine the capacitance of this system.

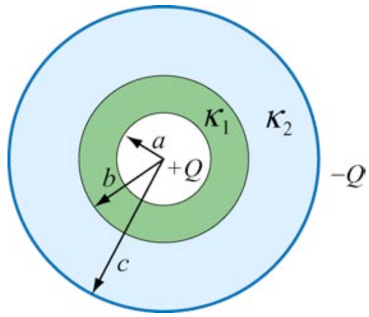


Figure 5.9.4 Spherical capacitor filled with dielectrics.

Solution: The system can be treated as two capacitors connected in series, since the total potential difference across the capacitors is the sum of potential differences across individual capacitors, $\Delta V = \Delta V_1 + \Delta V_2$. Each shell has the same magnitude charge $|Q|$. The charge on each capacitor is related to the potential difference by

$$\Delta V_i = \frac{Q}{C_i}.$$

Each individual capacitor, satisfies

$$C_i = \kappa_i C_{i,0}$$

where C_0 is the capacitance for a vacuum spherical capacitor of inner radius r_1 and outer radius r_2, which we calculated in Example 5.3,

$$C_{i,0} = 4\pi\varepsilon_0 \left(\frac{r_1 r_2}{r_2 - r_1} \right).$$

Therefore the capacitances are

$$C_1 = \kappa_1 4\pi\varepsilon_0 \left(\frac{ab}{b-a} \right)$$

$$C_2 = \kappa_2 4\pi\varepsilon_0 \left(\frac{bc}{c-b} \right).$$

The capacitance for the system therefore is

$$C = \frac{Q}{\Delta V_1 + \Delta V_2} = \frac{Q}{Q/C_1 + Q/C_2} = \frac{C_1 C_2}{C_1 + C_2}.$$

Using our results above we have that the capacitance of this system is given by

$$\begin{aligned}
C &= \frac{C_1 C_2}{C_1 + C_2} \\
&= \frac{\kappa_1 4\pi\varepsilon_0 \left(\dfrac{ab}{b-a}\right) \kappa_2 4\pi\varepsilon_0 \left(\dfrac{bc}{c-b}\right)}{\kappa_1 4\pi\varepsilon_0 \left(\dfrac{ab}{b-a}\right) + \kappa_2 4\pi\varepsilon_0 \left(\dfrac{bc}{c-b}\right)} \\
&= \frac{\kappa_1 4\pi\varepsilon_0 \left(\dfrac{ab}{b-a}\right) \kappa_2 4\pi\varepsilon_0 \left(\dfrac{bc}{c-b}\right)(b-a)(c-b)}{\kappa_1 4\pi\varepsilon_0 ab(c-b) + \kappa_2 4\pi\varepsilon_0 bc(b-a)}
\end{aligned}$$

Thus after some simplification we have that

$$\boxed{C = \frac{4\pi\varepsilon_0 \kappa_1 \kappa_2 abc}{\kappa_2 c(b-a) + \kappa_1 a(c-b)}}.$$

It is instructive to check the limit where $\kappa_1, \kappa_2 \to 1$. In this case, the above expression reduces to

$$C = \frac{4\pi\varepsilon_0 abc}{c(b-a) + a(c-b)} = \frac{4\pi\varepsilon_0 abc}{b(c-a)} = \frac{4\pi\varepsilon_0 ac}{(c-a)}$$

which agrees with Eq. (5.2.11) for a spherical capacitor of inner radius *a* and outer radius *c*.

5.9.3 Capacitor Connected to a Spring

Consider an air-filled parallel-plate capacitor with one plate connected to a spring having a force constant *k*, and another plate held fixed. The system rests on a table top as shown in Figure 5.9.5.

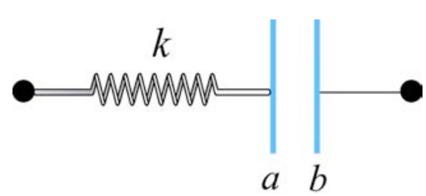


Figure 5.9.5 Capacitor connected to a spring.

If the charges placed on plates a and b are $+Q$ and $-Q$, respectively, how much does the spring expand?

Solution: The spring force $\vec{\mathbf{F}}_s$ acting on plate a is given by

$$\vec{\mathbf{F}}_s = -kx\,\hat{\mathbf{i}}\,.$$

Similarly, the electrostatic force $\vec{\mathbf{F}}_e$ due to the electric field created by plate b is

$$\vec{\mathbf{F}}_e = QE\,\hat{\mathbf{i}} = Q\left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{i}} = \frac{Q^2}{2A\varepsilon_0}\hat{\mathbf{i}}\,,$$

where A is the area of the plate. The charges on plate a cannot exert a force on itself, as required by Newton's third law. Thus, only the electric field due to plate b is considered. At equilibrium the two forces cancel and we have

$$kx = Q\left(\frac{Q}{2A\varepsilon_0}\right),$$

which gives

$$x = \frac{Q^2}{2kA\varepsilon_0}\,.$$

5.10 Conceptual Questions

1. The charges on the plates of a parallel-plate capacitor are of opposite sign, and they attract each other. To increase the plate separation, is the external work done positive or negative? What happens to the external work done in this process?

2. How does the stored energy change if the potential difference across a capacitor is tripled?

3. Does the presence of a dielectric increase or decrease the maximum operating voltage of a capacitor? Explain.

4. If a dielectric-filled capacitor is cooled down, what happens to its capacitance?

5.11 Additional Problems

5.11.1 Capacitors and Dielectrics

(a) A parallel-plate capacitor of area A and spacing d is filled with three dielectrics as shown in Figure 5.11.1. Each occupies 1/3 of the volume. What is the capacitance of this system? [*Hint:* Consider an equivalent system to be three parallel capacitors, and justify this assumption.] Show that you obtain the proper limits as the dielectric constants approach unity, $\kappa_i \to 1$.]

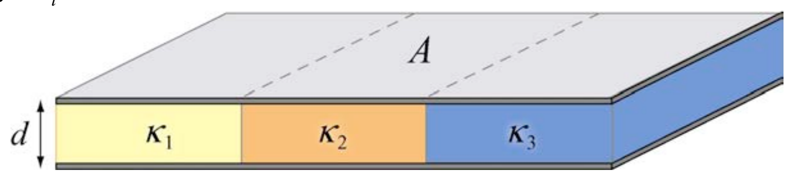


Figure 5.11.1

(b) This capacitor is now filled as shown in Figure 5.11.2. What is its capacitance? Use Gauss's law to find the field in each dielectric, and then calculate ΔV across the entire capacitor. Again, check your answer as the dielectric constants approach unity, $\kappa_i \to 1$. Could you have assumed that this system is equivalent to three capacitors in series?

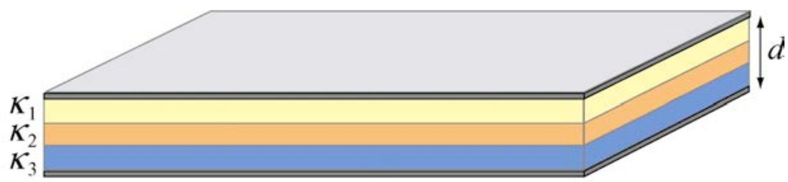


Figure 5.11.2

5.11.2 Gauss's Law in the Presence of a Dielectric

A solid conducting sphere with a radius R_1 carries a free charge Q and is surrounded by a concentric dielectric spherical shell with an outer radius R_2 and a dielectric constant κ_e. This system is isolated from other conductors and resides in air ($\kappa_e \approx 1$), as shown in Figure 5.11.3.

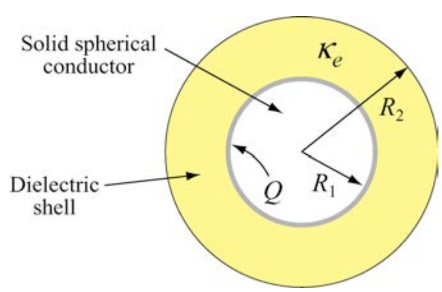


Figure 5.11.3

(a) Determine the displacement vector $\vec{\mathbf{D}}$ everywhere, *i.e.* its magnitude and direction in the regions $r < R_1$, $R_1 < r < R_2$ and $r > R_2$.

(b) Determine the electric field $\vec{\mathbf{E}}$ everywhere.

5.11.3 Gauss's Law and Dielectrics

A cylindrical shell of dielectric material has inner radius *a* and outer radius *b*, as shown in Figure 5.11.4.

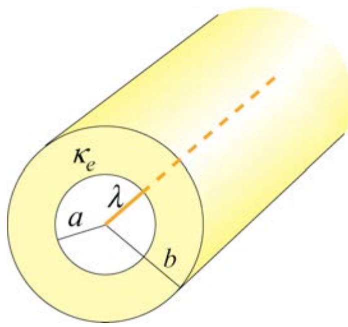


Figure 5.11.4

The material has a dielectric constant $\kappa_e = 10$. At the center of the shell there is a line charge running parallel to the axis of the cylindrical shell, with free charge per unit length λ.

(a) Find the electric field for: $r < a$, $a < r < b$ and $r > b$.

(b) What is the induced surface charge per unit length on the inner surface of the spherical shell? [Ans. $-9\lambda/10$.]

(c) What is the induced surface charge per unit length on the outer surface of the spherical shell? [Ans. $+9\lambda/10$.]

5.11.4 A Capacitor with a Dielectric

A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm^2, and a mica dielectric ($\kappa_e = 5.40$). At a 55 V potential difference, calculate

(a) the electric field strength in the mica; [Ans. 13.4 kV/m.]

(b) the magnitude of the free charge on the plates; [Ans. 6.16 nC.]

(c) the magnitude of the induced surface charge; [Ans. 5.02 nC.]

(d) the magnitude of the polarization $\vec{\mathbf{P}}$ [Ans. 520 nC/m^2.]

5.11.5 Force on the Plates of a Capacitor

The plates of a parallel-plate capacitor have area A and carry total charge $\pm Q$ (see Figure 5.12.6). We would like to show that these plates *attract* each other with a force given by $F = Q^2/(2\varepsilon_o A)$.

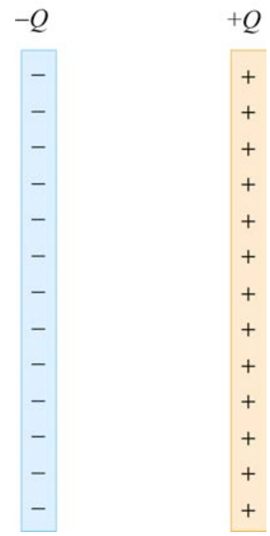


Figure 5.12.6

(a) Calculate the total force on the left plate due to the electric field of the right plate, using Coulomb's Law. Ignore fringing fields.

(b) If you pull the plates apart, against their attraction, you are doing work and *that work goes directly into creating additional electrostatic energy*. Calculate the force necessary to increase the plate separation from x to $x + dx$ by equating the work you do, $\vec{\mathbf{F}} \cdot d\vec{\mathbf{x}}$, to the increase in electrostatic energy, assuming that the electric energy density is $\varepsilon_0 E^2 / 2$, and that the charge Q remains constant.

(c) Using this expression for the force, show that the force per unit area (the *electrostatic stress*) acting on either capacitor plate is given by $\varepsilon_0 E^2 / 2$. This result is true for a conductor of any shape with an electric field $\vec{\mathbf{E}}$ at its surface.

(d) Atmospheric pressure is 14.7 lb/in^2, or 101,341 N/m^2. How large would E have to be to produce this force per unit area? [Ans. 151 MV/m. Note that Van de Graff accelerators can reach fields of 100 MV/m maximum before breakdown, so that electrostatic stresses are on the same order as atmospheric pressures in this extreme situation, but not much greater].

5.11.6 Energy Density in a Capacitor with a Dielectric

Consider the case in which a dielectric material with dielectric constant κ_e completely fills the space between the plates of a parallel-plate capacitor. Show that the energy density of the field between the plates is $u_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{D}} / 2$ by the following procedure:

(a) Write the expression $u_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{D}} / 2$ as a function of $\mathbf{E}$ and κ_e (i.e. eliminate $\vec{\mathbf{D}}$).

(b) Given the electric field and potential of such a capacitor with free charge q on it (problem 4-1a above), calculate the work done to charge up the capacitor from $q = 0$ to $q = Q$, the final charge.

(c) Find the energy density u_E.